SL Paper 2

Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 3 & 2 \\ 2 & 4 & \lambda \\ 3 & 7 & 3 \end{bmatrix}.$$

Suppose now that $\lambda = 1$ so consider the matrix

$$\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 7 & 3 \end{bmatrix}.$$

a.i. Find an expression for det(***A***) in terms of λ, simplifying your answer. [3]

a.ii. Hence show that ***A*** is singular when $\lambda = 1$ and find the other value of λ for which ***A*** is singular. [2]

b.i. Explain how it can be seen immediately that ***B*** is singular without calculating its determinant. [1]

b.ii. Determine the null space of ***B***. [4]

b.iii. Explain briefly how your results verify the rank-nullity theorem. [[N/A

c. Prove, using mathematical induction, that [7]

$$\boldsymbol{B}^n = 8^{n-2}\boldsymbol{B}^2 \text{ for } n \in \mathbb{Z}^+,\ n \geqslant 3.$$

Markscheme

a.i. det(***A***) $= \lambda(12 - 7\lambda) + 3(3\lambda - 6) + 2(14 - 12)$ ***M1A1***

$= 12\lambda - 7\lambda^2 + 9\lambda - 18 + 4$

$= -7\lambda^2 + 21\lambda - 14$ ***A1***

[??? marks]

a.ii. ***A*** is singular when $\lambda = 1$ because the determinant is zero ***R1***

Note: Do not award the ***R1*** if the determinant has not been obtained.

the other value is 2 ***A1***

[??? marks]

b.i. the third row is the sum of the first two rows ***A1***

[??? marks]

b.ii. the null space satisfies

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ***M1***

$x + 3y + 2z = 0$

$2x + 4y + z = 0$ ***(A1)***

$3x + 7y + 3z = 0$

the solution is (by GDC or otherwise)

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} \alpha$ where $\alpha \in \mathbb{R}$ ***M1A1***

[??? marks]

b.iii. the rank-nullity theorem for square matrices states that

rank of matrix + dimension of null space = number of columns ***A1***

here, rank = 2, dimension of null space = 1 and number of columns = 3 ***A1***

[??? marks]

c. first show that the result is true for $n = 3$

$\boldsymbol{B}^2 = \begin{bmatrix} 13 & 29 & 11 \\ 13 & 29 & 11 \\ 26 & 58 & 22 \end{bmatrix}$ ***A1***

$\boldsymbol{B}^3 = \begin{bmatrix} 104 & 232 & 88 \\ 104 & 232 & 88 \\ 208 & 464 & 176 \end{bmatrix}$ ***A1***

therefore $\boldsymbol{B}^3 = 8\boldsymbol{B}^2$ so true for $n = 3$ ***R1***

assume the result is true for $n = k$, that is $\boldsymbol{B}^k = 8^{k-2}\boldsymbol{B}^2$ ***M1***

consider $\boldsymbol{B}^{k+1} = 8^{k-2}\boldsymbol{B}^2$ ***M1***

$= 8^{k-2}8\boldsymbol{B}^2$

$= 8^{k-1}\boldsymbol{B}^2$ ***A1***

therefore, true for $n = k \Rightarrow$ true for $n = k + 1$ and since the result is true for $n = 3$, it is true for $n \geqslant 3$ ***R1***

[7 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]
c. [N/A]

The hyperbola with equation $x^2 - 4xy - 2y^2 = 3$ is rotated through an acute anticlockwise angle α about the origin.

a. The point $(x,\ y)$ is rotated through an anticlockwise angle α about the origin to become the point $(X,\ Y)$. Assume that the rotation can be represented by [3]

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show, by considering the images of the points $(1,\ 0)$ and $(0,\ 1)$ under this rotation that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

b.i. By expressing $(x,\ y)$ in terms of $(X,\ Y)$, determine the equation of the rotated hyperbola in terms of X and Y. [3]

b.ii. Verify that the coefficient of XY in the equation is zero when $\tan\alpha = \frac{1}{2}$. [3]

b.iii. Determine the equation of the rotated hyperbola in this case, giving your answer in the form $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$. [3]

b.iv. Hence find the coordinates of the foci of the hyperbola prior to rotation. [5]

Markscheme

a. consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ ***(M1)***

the image of $(1,\ 0)$ is $(\cos\alpha,\ \sin\alpha)$ ***A1***

therefore $a = \cos\alpha,\ c = \sin\alpha$ ***AG***

consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$

the image of $(0,\ 1)$ is $(-\sin\alpha,\ \cos\alpha)$ ***A1***

therefore $b = -\sin\alpha,\ d = \cos\alpha$ ***AG***

[3 marks]

b.i. $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

or $x = X\cos\alpha + Y\sin\alpha,\ y = -X\sin\alpha + Y\cos\alpha$ ***A1***

substituting in the equation of the hyperbola, ***M1***

$(X\cos\alpha + Y\sin\alpha)^2 - 4(X\cos\alpha + Y\sin\alpha)(-X\sin\alpha + Y\cos\alpha)$

$-2(-X\sin\alpha + Y\cos\alpha)^2 = 3$ ***A1***

$X^2(\cos^2\alpha - 2\sin^2\alpha + 4\sin\alpha\cos\alpha) +$

$XY(2\sin\alpha\cos\alpha - 4\cos^2\alpha + 4\sin^2\alpha + 4\sin\alpha\cos\alpha) +$

$Y^2(\sin^2\alpha - 2\cos^2\alpha - 4\sin\alpha\cos\alpha) = 3$

[??? marks]

b.ii. when $\tan\alpha = \frac{1}{2},\ \sin\alpha = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \frac{2}{\sqrt{5}}$ ***A1***

the XY term $= 6\sin\alpha\cos\alpha - 4\cos^2\alpha + 4\sin^2\alpha$ ***M1***

$= 6 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} - 4 \times \frac{4}{5} + 4 \times \frac{1}{5}\left(\frac{12}{5} - \frac{16}{5} + \frac{4}{5}\right)$ ***A1***

$= 0$ ***AG***

[??? marks]

b.iii. the equation of the rotated hyperbola is

$2X^2 - 3Y^2 = 3$ ***M1A1***

$\frac{X^2}{\left(\sqrt{\frac{3}{2}}\right)^2} - \frac{Y^2}{(1)^2} = 1$ ***A1***

$\left(\text{accept } \frac{X^2}{\frac{3}{2}} - \frac{Y^2}{1} = 1\right)$

[??? marks]

b.iv. the coordinates of the foci of the rotated hyperbola

are $\left(\pm\sqrt{\frac{3}{2}+1},\ 0\right) = \left(\pm\sqrt{\frac{5}{2}},\ 0\right)$ ***M1A1***

the coordinates of the foci prior to rotation were given by

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \pm\sqrt{\frac{5}{2}} \\ 0 \end{bmatrix}$$

M1A1

$\begin{bmatrix} \pm\sqrt{2} \\ \mp\frac{1}{\sqrt{2}} \end{bmatrix}$ ***A1***

[??? marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]
b.iv. [N/A]

S is defined as the set of all 2×2 non-singular matrices. A and B are two elements of the set S.

a. (i) Show that $(A^T)^{-1} = (A^{-1})^T$. [8]

(ii) Show that $(AB)^T = B^T A^T$.

b. A relation R is defined on S such that A is related to B if and only if there exists an element X of S such that $XAX^T = B$. Show that R is an equivalence relation. [8]

Markscheme

a. (i) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ ***M1***

$(A^T)^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ (which exists because $ad - bc \neq 0$) ***A1***

$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ***M1***

$(A^{-1})^T = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ ***A1***

hence $(A^T)^{-1} = (A^{-1})^T$ as required ***AG***

(ii) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ ***M1***

$(AB)^T = \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}$ ***A1***

$B^T = \begin{pmatrix} e & g \\ f & h \end{pmatrix}$ $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ ***M1***

$B^TA^T = \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}$ ***A1***

hence $(AB)^T = B^TA^T$ ***AG***

b. R is reflexive since $I \in S$ and $IAI^T = A$ ***A1***

$XAX^T = B \Rightarrow A = X^{-1}B(X^T)^{-1}$ ***M1A1***

$\Rightarrow A = X^{-1}B(X^{-1})^T$ from a (i) ***A1***

which is of the correct form, hence symmetric ***AG***

$ARB \Rightarrow XAX^T = B$ and $BRC = YBY^T = C$ ***M1***

Note: Allow use of X rather than Y in this line.

$\Rightarrow YXAX^TY^T = YBY^T = C$ ***M1A1***

$\Rightarrow (YX)A(YX)^T = C$ from a (ii) ***A1***

this is of the correct form, hence transitive

hence R is an equivalence relation ***AG***

Examiners report

a. Part a) was successfully answered by the majority of candidates..

b. There were some wholly correct answers seen to part b) but a number of candidates struggled with the need to formally explain what was required.

The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 11 & 8 \\ 1 & 3 & 4 & \lambda \\ \lambda & 5 & 7 & 6 \end{pmatrix}$.

(a) Given that $\lambda = 2$, $\boldsymbol{B} = \begin{pmatrix} 2 \\ 4 \\ \mu \\ 3 \end{pmatrix}$ and $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$,

(i) find the value of μ for which the equations defined by $\boldsymbol{AX} = \boldsymbol{B}$ are consistent and solve the equations in this case;

(ii) define the rank of a matrix and state the rank of $\boldsymbol{A}$.

(b) Given that $\lambda = 1$,

(i) show that the four column vectors in ***A*** form a basis for the space of four-dimensional column vectors;

(ii) express the vector $\begin{pmatrix} 6 \\ 28 \\ 12 \\ 15 \end{pmatrix}$ as a linear combination of these basis vectors.

Markscheme

(a) (i) using row reduction, ***M1***

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 3 & 8 & 11 & 8 & 4 \\ 1 & 3 & 4 & 2 & \mu \\ 2 & 5 & 7 & 6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 0 & 2 & 2 & -4 & -2 \\ 0 & 1 & 1 & -2 & \mu-2 \\ 0 & 1 & 1 & -2 & -1 \end{pmatrix} \quad \textit{(A2)}$$

for consistency,

$\mu - 2 = -1$ ***(M1)***

$\mu = 1$ ***A1***

put $z = \alpha,\ t = \beta$ ***M1***

$y = -1 - \alpha + 2\beta;\ x = 4 - \alpha - 8\beta$ ***A1A1***

(ii) the rank of a matrix is the number of independent rows (or columns) ***A1***

rank $(\boldsymbol{A}) = 2$ ***A1***

[10 marks]

(b) (i) det $(\boldsymbol{A}) = 2$ ***(M1)A1***

since det $(\boldsymbol{A}) \neq 0$, the vectors form a basis ***R1***

(ii) let $\begin{pmatrix} 6 \\ 28 \\ 12 \\ 15 \end{pmatrix} = a\begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 8 \\ 3 \\ 5 \end{pmatrix} + c\begin{pmatrix} 3 \\ 11 \\ 4 \\ 7 \end{pmatrix} + d\begin{pmatrix} 4 \\ 8 \\ 1 \\ 6 \end{pmatrix}$ ***M1***

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 11 & 8 \\ 1 & 3 & 4 & 1 \\ 1 & 5 & 7 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

it follows that

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 11 & 8 \\ 1 & 3 & 4 & 1 \\ 1 & 5 & 7 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 28 \\ 12 \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

therefore

$a = 2$ ***A1***

$b = 1$ ***A1***

$c = 2$ ***A1***

$d = -1$ ***A1***

$$\begin{pmatrix} 6 \\ 28 \\ 12 \\ 15 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \\ 3 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 11 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \\ 1 \\ 6 \end{pmatrix}$$

[8 marks]

Examiners report

[N/A]

The set of all permutations of the list of the integers $1,\ 2,\ 3\ \ldots\ n$ is a group, S_n, under the operation of composition of permutations.

Each element of S_4 can be represented by a 4×4 matrix. For example, the cycle $(1\ 2\ 3\ 4)$ is represented by the matrix

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ acting on the column vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

a. (i) Show that the order of S_n is $n!$; [9]

(ii) List the 6 elements of S_3 in cycle form;

(iii) Show that S_3 is not Abelian;

(iv) Deduce that S_n is not Abelian for $n \geqslant 3$.

b. (i) Write down the matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ representing the permutations $(1\ 2),\ (2\ 3)$, respectively; [7]

(ii) Find $\boldsymbol{M}_1\boldsymbol{M}_2$ and state the permutation represented by this matrix;

(iii) Find $\det(\boldsymbol{M}_1),\ \det(\boldsymbol{M}_2)$ and deduce the value of $\det(\boldsymbol{M}_1\boldsymbol{M}_2)$.

c. (i) Use mathematical induction to prove that [8]

$(1\ n)(1\ n\ -1)(1\ n-2)\ldots(1\ 2) = (1\ 2\ 3\ldots n)\ n \in \mathbb{Z}^+,\ n > 1.$

(ii) Deduce that every permutation can be written as a product of cycles of length 2.

Markscheme

a. (i) 1 has n possible new positions; 2 then has $n-1$ possible new positions…

n has only one possible new position ***R1***

the number of possible permutations is $n \times (n-1) \times \ldots \times 2 \times 1$ ***R1***

$= n!$ ***AG***

Note: Give no credit for simply stating that the number of permutations is $n!$

(ii) $(1)(2)(3);\ (1\ 2)(3);\ (1\ 3)(2);\ (2\ 3)(1);\ (1\ 2\ 3);\ (1\ 3\ 2)$ ***A2***

Notes: *A1* for 4 or 5 correct.

If single bracket terms are missing, do not penalize.

Accept e in place of the identity.

(iii) attempt to compare $\pi_1 \circ \pi_2$ with $\pi_2 \circ \pi_1$ for two permutations ***M1***

for example $(1\ 2)(1\ 3) = (1\ 3\ 2)$ ***A1***

but $(1\ 3)(1\ 2) = (1\ 2\ 3)$ ***A1***

hence S_3 is not Abelian ***AG***

(iv) S_3 is a subgroup of S_n, ***R1***

so S_n contains non-commuting elements ***R1***

$\Rightarrow S_n$ is not Abelian for $n \geqslant 3$ ***AG***

[9 marks]

b.

(i) $\boldsymbol{M}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \boldsymbol{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ***A1A1***

(ii) $\boldsymbol{M}_1\boldsymbol{M}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ***A1***

this represents $(1\ 3\ 2)$ ***A1***

(iii) by, for example, interchanging a pair of rows (***M1***)

$\det(\boldsymbol{M}_1) = \det(\boldsymbol{M}_2) = -1$ ***A1***

then $\det(\boldsymbol{M}_1\boldsymbol{M}_2) = (-1) \times (-1) = 1$ ***A1***

[7 marks]

c. (i) let $\mathrm{P}(n)$ be the proposition that

$(1\ n)(1\ n-1)(1\ n-2)\ldots(1\ 2) = (1\ 2\ 3\ldots n)\ n \in \mathbb{Z}^+$

the statement that $\mathrm{P}(2)$ is true *eg* $(1\ 2) = (1\ 2)$ ***A1***

assume $\mathrm{P}(k)$ is true for some k ***M1***

consider $(1\ k+1)(1\ k)(1\ k-1)(1\ k-2)\ldots(1\ 2)$

$= (1\ k+1)(1\ 2\ 3\ldots k)$ ***M1***

then the composite permutation has the following effect on the first $k+1$ integers: $1 \to 2,\ 2 \to 3 \ldots k-1 \to k,\ k \to 1 \to k+1,\ k+1 \to 1$ ***A1***

this is $(1\ 2\ 3\ldots k\ k+1)$ ***A1***

hence the assertion is true by induction ***AG***

(ii) every permutation is a product of cycles ***R1***

generalizing the result in (i) ***R1***

every cycle is a product of cycles of length 2 ***R1***

hence every permutation can be written as a product of cycles of length 2 ***AG***

[8 marks]

Examiners report

a. In part (a)(i), many just wrote down $n!$ without showing how this arises by a sequential choice process. Part (ii) was usually correctly answered, although some gave their answers in the unwanted 2-dimensional form. Part (iii) was often well answered, though some candidates failed to realise that they need to explicitly evaluate the product of two elements in both orders.

b. Part (b) was often well answered. A number of candidates found 2×2 matrices – this gained no marks.

c. Nearly all candidates knew how to approach part (c)(i), but failed to be completely convincing. Few candidates seemed to know that every permutation can be written as a product of non-overlapping cycles, as the first step in part (ii).

---

a. Given that the elements of a 2×2 symmetric matrix are real, show that [11]

(i) the eigenvalues are real;

(ii) the eigenvectors are orthogonal if the eigenvalues are distinct.

b. The matrix $\boldsymbol{A}$ is given by [7]

$$\boldsymbol{A} = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

c. The ellipse E has equation $\boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X} = 24$ where $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{A}$ is as defined in part (b). [7]

(i) Show that E can be rotated about the origin onto the ellipse E' having equation $2x^2 + 3y^2 = 6$.

(ii) Find the acute angle through which E has to be rotated to coincide with E' .

Markscheme

a. (i) let $\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ ***(M1)***

the eigenvalues satisfy

$\det(\boldsymbol{M} - \lambda\boldsymbol{I}) = 0$ ***(M1)***

$(a - \lambda)(c - \lambda) - b^2 = 0$ ***(A1)***

$\lambda^2 - \lambda(a + c) + ac - b^2 = 0$ ***A1***

discriminant $= (a + c)^2 - 4(ac - b^2)$ ***M1***

$= (a - c)^2 + 4b^2 \geq 0$ ***A1***

this shows that the eigenvalues are real ***AG***

(ii) let the distinct eigenvalues be λ_1, λ_2 , with eigenvectors $\boldsymbol{X}_1, \boldsymbol{X}_2$

then

$\lambda_1\boldsymbol{X}_1 = \boldsymbol{M}\boldsymbol{X}_1$ and $\lambda_2\boldsymbol{X}_2 = \boldsymbol{M}\boldsymbol{X}_1$ ***M1***

transpose the first equation and postmultiply by $\boldsymbol{X}_2$ to give

$\lambda_1\boldsymbol{X}_1^T\boldsymbol{X}_2 = \boldsymbol{X}_1^T\boldsymbol{M}\boldsymbol{X}_2$ ***A1***

premultiply the second equation by $\boldsymbol{X}_1^T$

$\lambda_2 \boldsymbol{X}_1^T \boldsymbol{X}_2 = \boldsymbol{X}_1^T \boldsymbol{M} \boldsymbol{X}_2$ ***A1***

it follows that

$(\lambda 1 - \lambda_2)\boldsymbol{X}_1^T \boldsymbol{X}_2 = 0$ ***A1***

since $\lambda 1 \neq \lambda_2$, it follows that $\boldsymbol{X}_1^T \boldsymbol{X}_2 = 0$ so that the eigenvectors are orthogonal ***R1***

[11 marks]

b. the eigenvalues satisfy $\begin{vmatrix} 11-\lambda & \sqrt{3} \\ \sqrt{3} & 9-\lambda \end{vmatrix} = 0$ ***M1A1***

$\lambda^2 - 20\lambda + 96 = 0$ ***A1***

$\lambda = 8, 12$ ***A1***

first eigenvector satisfies

$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ***M1***

$\begin{pmatrix} x \\ y \end{pmatrix} = \text{(any multiple of)} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$ ***A1***

second eigenvector satisfies

$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \text{(any multiple of)} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ ***A1***

[7 marks]

c. (i) consider the rotation in which (x, y) is transformed onto (x', y') defined by

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ so that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$ ***M1A1***

the ellipse E becomes

$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 24$ ***M1A1***

$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 24$ ***A1***

$2(x')^2 + 3(y')^2 = 6$ ***AG***

(ii) the angle of rotation is given by $\cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2}$ ***M1***

since a rotational matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

so $\theta = 60^\circ$ (anticlockwise) ***A1***

[7 marks]

Examiners report

a. [N/A]

b. [N/A]

c. [N/A]

The function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ is defined by $\boldsymbol{X}\mapsto\boldsymbol{AX}$, where $\boldsymbol{X}=\begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a , b , c , d are all non-zero.

Consider the group $\{S, +_m\}$ where $S=\{0,1,2\ldots m-1\}$, $m\in\mathbb{N}$, $m\geq 3$ and $+_m$ denotes addition modulo m .

A.a Show that f is a bijection if $\boldsymbol{A}$ is non-singular. [7]

A.b Suppose now that $\boldsymbol{A}$ is singular. [5]

(i) Write down the relationship between a , b , c , d .

(ii) Deduce that the second row of $\boldsymbol{A}$ is a multiple of the first row of $\boldsymbol{A}$.

(iii) Hence show that f is not a bijection.

B.a Show that $\{S, +_m\}$ is cyclic for all $\boldsymbol{m}$. [3]

B.b Given that m is prime, [7]

(i) explain why all elements except the identity are generators of $\{S, +_m\}$;

(ii) find the inverse of x , where $\boldsymbol{x}$ is any element of $\{S, +_m\}$ apart from the identity;

(iii) determine the number of sets of two distinct elements where each element is the inverse of the other.

B.c Suppose now that $m=ab$ where a , b are unequal prime numbers. Show that $\{S, +_m\}$ has two proper subgroups and identify them. [3]

Markscheme

A.a recognizing that the function needs to be injective and surjective ***R1***

Note: Award ***R1*** if this is seen anywhere in the solution.

injective:

let $\boldsymbol{U},\boldsymbol{V}\in^\circ\times^\circ$ be 2-D column vectors such that $\boldsymbol{AU}=\boldsymbol{AV}$ ***M1***

$\boldsymbol{A}^{-1}\boldsymbol{AU}=\boldsymbol{A}^{-1}\boldsymbol{AV}$ ***M1***

$\boldsymbol{U}=\boldsymbol{V}$ ***A1***

this shows that f is injective

surjective:

let $W\in^\circ\times^\circ$ ***M1***

then there exists $\boldsymbol{Z}=\boldsymbol{A}^{-1}\boldsymbol{W}\in^\circ\times^\circ$ such that $\boldsymbol{AZ}=\boldsymbol{W}$ ***M1A1***

this shows that f is surjective

therefore f is a bijection ***AG***

[7 marks]

A.b (i) the relationship is $ad=bc$ ***A1***

(ii) it follows that $\frac{c}{a}=\frac{d}{b}=\lambda$ so that $(c,d)=\lambda(a,b)$ ***A1***

(iii) **EITHER**

let $\boldsymbol{W} = \begin{bmatrix} p \\ q \end{bmatrix}$ be a 2-D vector

then $\boldsymbol{AW} = \begin{bmatrix} a & b \\ \lambda a & \lambda b \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$ ***M1***

$= \begin{bmatrix} ap + bq \\ \lambda(ap + bq) \end{bmatrix}$ ***A1***

the image always satisfies $y = \lambda x$ so f is not surjective and therefore not a bijection ***R1***

OR

consider

$$\begin{bmatrix} a & b \\ \lambda a & \lambda b \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} ab \\ \lambda ab \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ \lambda a & \lambda b \end{bmatrix} \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} ab \\ \lambda ab \end{bmatrix}$$

this shows that f is not injective and therefore not a bijection ***R1***

[5 marks]

B.a. the identity element is 0 ***R1***

consider, for $1 \le r \le m$,

using 1 as a generator ***M1***

1 combined with itself r times gives r and as r increases from 1 to m, the group is generated ending with 0 when $r = m$ ***A1***

it is therefore cyclic ***AG***

[3 marks]

B.b. (i) by Lagrange the order of each element must be a factor of m and if m is prime, its only factors are 1 and m ***R1***

since 0 is the only element of order 1, all other elements are of order m and are therefore generators ***R1***

(ii) since $x +_m (m - x) = 0$ ***(M1)***

the inverse of $\boldsymbol{x}$ is $(m - x)$ ***A1***

(iii) consider

element	inverse
1	$m-1$
2	$m-2$
.	.
.	.
.	.
$\frac{1}{2}(m-1)$	$\frac{1}{2}(m+1)$

M1A1

there are $\frac{1}{2}(m - 1)$ inverse pairs ***A1 N1***

Note: Award ***M1*** for an attempt to list the inverse pairs, ***A1*** for completing it correctly and ***A1*** for the final answer.

[7 marks]

B.c. since a, b are unequal primes the only factors of m are a and b

there are therefore only subgroups of order a and b ***R1***

they are

$\{0, a, 2a, \ldots, (b-1)a\}$ ***A1***

$\{0, b, 2b, \ldots, (a-1)b\}$ ***A1***

[3 marks]

Examiners report

A.a. This proved to be a difficult question for some candidates. Most candidates realised that they had to show that the function was both injective and surjective but many failed to give convincing proofs. Some candidates stated, incorrectly, that f was injective because $\boldsymbol{AX}$ is uniquely defined, not realising that they had to show that $\boldsymbol{AX} = \boldsymbol{AY} \Rightarrow \boldsymbol{X} = \boldsymbol{Y}$.

A.b. Solutions to (b) were disappointing with many candidates failing to realise that they had either to show that $\boldsymbol{AX}$ was confined to a subset of $\mathbb{R} \times \mathbb{R}$ or that two distinct vectors had the same image under f.

B.a. This question was well answered in general with solutions to (c) being the least successful.

B.b. This question was well answered in general with solutions to (c) being the least successful.

B.c. This question was well answered in general with solutions to (c) being the least successful.

---

a. By considering the points $(1,\ 0)$ and $(0,\ 1)$ determine the 2×2 matrix which represents [5]

(i) an anticlockwise rotation of θ about the origin;

(ii) a reflection in the line $y = (\tan\theta)x$.

b. Determine the matrix A which represents a rotation from the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to the direction $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. [2]

c. A triangle whose vertices have coordinates $(0,\ 0)$, $(3,\ 1)$ and $(1,\ 5)$ undergoes a transformation represented by the matrix $A^{-1}XA$, where X is the matrix representing a reflection in the x-axis. Find the coordinates of the vertices of the transformed triangle. [6]

d. The matrix $B = A^{-1}XA$ represents a reflection in the line $y = mx$. Find the value of m. [6]

Markscheme

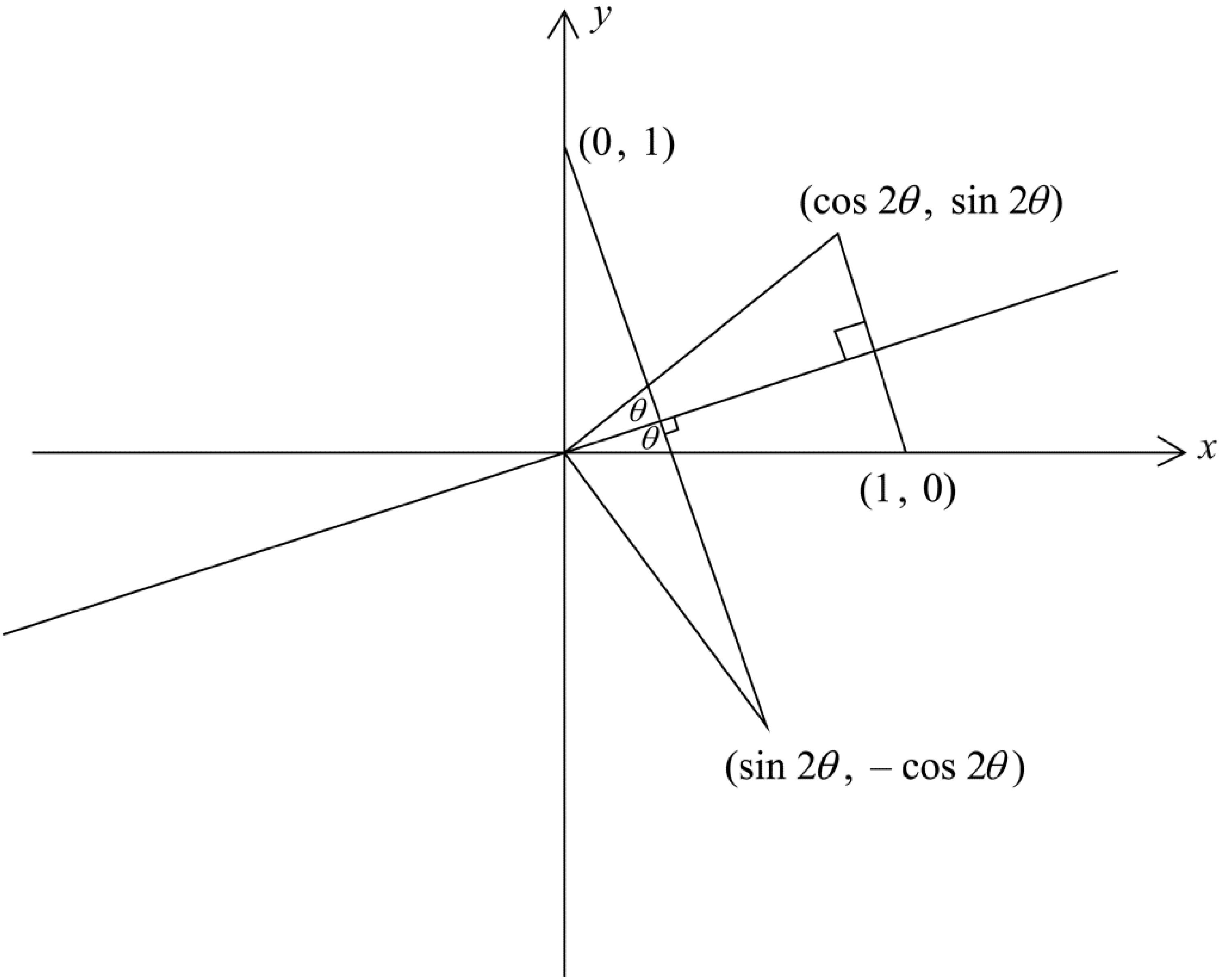
a. (i) under an anti-clockwise rotation of θ

$(1,\ 0) \to (\cos\theta,\ \sin\theta)$

$(0,\ 1) \to (-\sin\theta,\ \cos\theta)$ ***M1***

rotation matrix is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ***A1***

(ii)



M1

under a reflection in the line $y = (\tan\theta)x$

$(1,\ 0) \rightarrow (\cos 2\theta,\ \sin 2\theta)$

$(0,\ 1) \rightarrow (\sin 2\theta,\ -\cos 2\theta)$ ***M1***

matrix for reflection in the line $y = (\tan\theta)x$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ ***A1***

b. in this case $\tan\theta = 3$ (***M1)***

$\Rightarrow \sin\theta = \frac{3}{\sqrt{10}}$

hence rotation matrix is $\begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$ ***A1***

c. $A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$ ***(A1)***

$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ***(A1)***

$\Rightarrow A^{-1}XA = \begin{pmatrix} -\frac{8}{10} & -\frac{6}{10} \\ -\frac{6}{10} & \frac{8}{10} \end{pmatrix}$ ***(M1)A1***

$\Rightarrow A^{-1}XA(G) = \begin{pmatrix} -\frac{8}{10} & -\frac{6}{10} \\ -\frac{6}{10} & \frac{8}{10} \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -3 & -\frac{19}{5} \\ 0 & -1 & \frac{17}{5} \end{pmatrix}$ ***(M1)***

hence coordinates are $(0,\ 0)$, $(-3,\ -1)$ and $\left(-\frac{19}{5},\ \frac{17}{5}\right)$ ***A1***

d. $B = \begin{pmatrix} -\frac{8}{10} & -\frac{6}{10} \\ -\frac{6}{10} & \frac{8}{10} \end{pmatrix}$

the matrix for the reflection in the line $y = (\tan\theta)x$ is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$\cos 2\theta = -\frac{4}{5},\ \sin 2\theta = -\frac{3}{5}$ ***(A1)(A1)***

$\cos 2\theta = 2\cos^2\theta - 1$ ***(M1)***

$\Rightarrow 2\cos^2\theta = \frac{2}{10}$

$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{10}}$ ***(A1)***

$\Rightarrow \cos\theta = -\frac{1}{\sqrt{10}}$ and $\sin\theta = \frac{3}{\sqrt{10}}$ ***(A1)***

$\Rightarrow \tan\theta = -3$

$\Rightarrow m = -3$ ***A1***

Examiners report

a. This proved to be a more challenging question for many candidates. In part a) many candidates appeared to not know how to find the matrices and for those who attempted to find them, arithmetic errors were common. A number of wholly correct solutions to parts b), c) and d) were seen, but many candidates seemed unfamiliar with this style of question and made errors or simply gave up part way through the process.

b. This proved to be a more challenging question for many candidates. In part a) many candidates appeared to not know how to find the matrices and for those who attempted to find them, arithmetic errors were common. A number of wholly correct solutions to parts b), c) and d) were seen, but many candidates seemed unfamiliar with this style of question and made errors or simply gave up part way through the process.

c. This proved to be a more challenging question for many candidates. In part a) many candidates appeared to not know how to find the matrices and for those who attempted to find them, arithmetic errors were common. A number of wholly correct solutions to parts b), c) and d) were seen, but many candidates seemed unfamiliar with this style of question and made errors or simply gave up part way through the process.

d. This proved to be a more challenging question for many candidates. In part a) many candidates appeared to not know how to find the matrices and for those who attempted to find them, arithmetic errors were common. A number of wholly correct solutions to parts b), c) and d) were seen, but many candidates seemed unfamiliar with this style of question and made errors or simply gave up part way through the process.